Doundary Value Problems (BVP's) So har we're been solving IVP's like  $P \qquad S \qquad y'' = F(y', y, t) \\ C \qquad (to) = \alpha , y'(to) = \beta$  $(\gamma = f(\gamma t))$ ~ y(to) = y In contrast, BVP's are differential equations with conditions at two different points. For example: (y'' = f(t, y, y') $(y(a)=\alpha, y(b)=\beta$ Remark ? In general, we need 2 values to solve a 2<sup>nd</sup> order ODE Example : y" = - y has the general solution

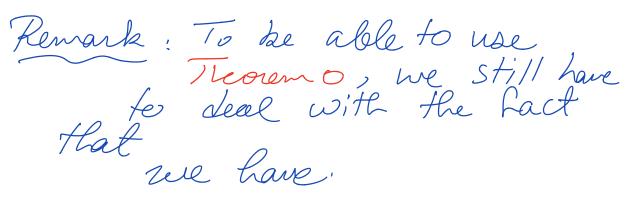
y(t) = Asint + B cost Then, given say y(0) & y'(0) as en ZVP's we an solve for A,B But we can also solve for A,B y(0) ( y(T/2) (Sary) given (as in BVP's) Lassunia ne get a consistent In general, we need numerical methods for solving BVP's ! We will soon consider these, but First let's worry about existence of solutions to BVP's Theoremo Theorem : The BVP  $\int x'' = F(t, x)$ (x(0) = 0, x(1) = 0

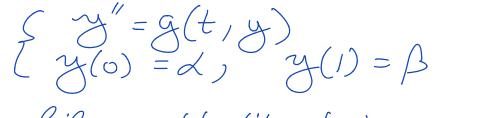
has a unique solh if is is cont., non-neg., bounded in [0,1] × R Example: Show that  $\int \alpha'' = (5\alpha + \sin 3\alpha)e^t$  $\int \chi(0) = \chi(1) = 0$ has a runique solin: Solin:  $f(t, x) = (5x + 3i)e^{t}$  $=) \frac{\partial F}{\partial x} = (5 + 3 \cos 3x) e^{T}$ which is cont., non-neg(unly?), & bounded (by 8e) on LO, I x R =) we have a unique sol'n by the theorem.

Applying the theorem to more general BVPs, like  $\begin{cases} x'' = F(t, x) \\ x(a) = \alpha, \quad y(b) = \beta \end{cases}$ We'll get there in 2 steps: 1st Theorem on two-point BVP's Consider: (2)(y'=q(t,y)) $(\mathcal{D}_{\mathcal{C}} \times \mathcal{A}' = \mathcal{F}(t, x))$  $\int \mathcal{I}(a) = 2$ ,  $\mathcal{I}(b) = \beta$   $(\mathcal{Y}(b) = a, \mathcal{Y}(i) = \beta$ where g(p,q) = (b-a)<sup>2</sup> (F(a+(b-a)p,q) \* If y solves 2, then  $\chi(t) = \chi(\frac{t-a}{b-a})$  solves () or solves D, then \* If  $\gamma(t) = \chi(a + (b - a)t)$  solves (2)

Proof: Simply check that the change of variables works (see book),

Remark: The above theorem simply semaps the interval (a, B to [0,1]





while we'd like to have d=B=0

Solas Just subtract a linear function!

2nd Theorem on two-point BVP's Consider: (3)(Z'=h(t,Z)) $\mathcal{D}(\mathbf{y}''=\mathbf{g}(t,\mathbf{y})$ (Z(0)=0) Z(1)=1 $(y(0) = a, y(1) = \beta$ where h(p,q) = g(p,q+d+ (B-d)p) \* If Z solves (3), then  $\mathcal{Y}(t) = \mathcal{Z}(t) + \alpha + (\beta - \alpha)t \quad solves(2)$ \* If y solves Q, then  $Z(t) = y(t) - [\alpha + (\beta - \alpha)t]$  solves (3)

Troof: Check. Example on how to use the 3 theorems together Convert the two-point BVP to an equivalent one with a boundary values on ) أروكر

 $\int x'' = x^2 + 3 - t^2 + xt$ (x(3) = 7, x(5) = 9CSolin =(1) By the 1theorem: (\*) is equivalent to  $\begin{cases} y'' = g(t,y) \\ y(0) = 7, y(1) = 9 \end{cases}$ with g(t, y) = 4f(3+2t, y) $(b-a)^2$  a+(b-a)t $SO G(t,y) = 4 \left[ y^{2} + 3 - (3+2t)^{2} + \chi(3+2t) \right]$ 

(2) By the 2nd theorem (\*) is now equivalent to z'=h(t,z)Z(0) = 0 , Z(1) = 0with h(t, z) = g(t, z + 7 + 2t) $= 4 \int (z + 7 + 2t)^{2} + 3 - (3 + 2t)^{2} + (z + 7 + 2t)^{2} + (3 + 2t)^{2} + ($ So, to solve for a, we an first

solve for Z, then substitute

y(t) = Z(t) + 7 + 2t

and  $\chi(t) = \chi(\frac{t-3}{2})$ 

/heorem Let Ats be cont's on [0,] x R with  $|f(t,s_1) - f(t,s_2)| \le k |s_1 - s_2|$ 1 lipschij where K<8 Then  $\int x'' = F(t, \alpha)$  $\int \chi(0) = 0 = \mathcal{I}(I)$ has a unique solo in C[0, ]]

Example 5  $(x''=2e^{t\cos 2})$ (x(0)=x(1)=0

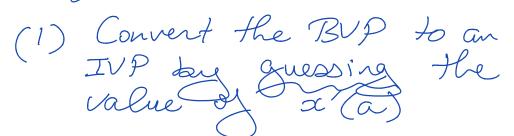
•  $f(t, x) = 2e^{t cosc}$ 

 $) \frac{\partial F}{\partial n} = 2t \sin x e^{t \cos x}$ => | 2F | 5 2c on [0] XR

F(t,x) is Lipschit in x on [o,1]xR with const 2e < & =) we have a unique solly by the Heorem above,



 $\begin{cases} x'' = f(t, x, x') \\ \chi(a) = a, \chi(b) = \beta \end{cases}$ KoughIdea :

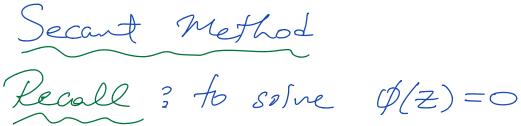


(2) Integrate the equation

(3) Check if  $x(b) = \beta \cdot If$   $x(b) \neq \beta \cdot guess x'(a)$ and thy again

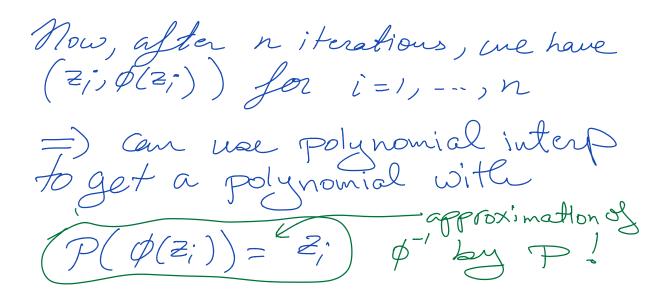
So now :  $(*) \begin{cases} \pi'' = f(t, \pi, \pi') \\ \pi(a) = \pi, \qquad \pi'(a) = z \\ \pi$ this is a guess het the soln of (\*) be  $\chi_z$ . We'd like  $\chi_z$ : B, so we define  $(\phi(z) = \chi_z(b) - \beta)$ The error from our guess Our goal is to get p(z) = 0. This is "just" a nonlinear equation that we need to solve =) can use Newton's method, secant method, bisection method, etc ----

OTOH, we don't explicitly have  $\phi(z)$ , Need to solve an IVP numerically traduction of  $\phi(z)$ 



ve run the iteration

 $\left(Z_{n} = Z_{n-1} - \left(\frac{Z_{n-1} - Z_{n-2}}{\phi(Z_{n-1}) - \phi(Z_{n-2})}\right)\phi(Z_{n-1})\right)$ 



now let  $Z_{n+1} = \mathcal{P}(0)$  $\mathcal{M}_{a} = \frac{\mathcal{P}(z_{n+1}) = 0}{\mathcal{M}_{a}}$   $\mathcal{M}_{a} = \frac{\mathcal{P}(z_{n+1})}{\mathcal{P}(z_{n+1})} = 0$   $\mathcal{M}_{a} = \frac{\mathcal{P}(z_{n+1})}{\mathcal{P}(z_{n+1})} = 0$ lo Summarize: • Want to solve S x'' = F(t, x, x') $\int x(a) = \lambda, x(b) = \beta$ 

Instead, we solve  $\int x'' = F(t, x, x')$   $\int \chi(a) = d, \chi(a) = Z,$ • This give a sol'n Xz & a boundary value Zz (b) So our error is  $( p(z_i) = \alpha_{z_i}(b) - \beta )$ 

• Do it agan with Z2 to get  $(\mathcal{P}(z_2) = \mathcal{Z}_{z_2}(b) - \beta)$ 

Now run the secont method  $= Z_{n} = Z_{n-1} - \left(\frac{\phi(z_{n}) - \phi(z_{n-2})}{z_{n-1} - z_{n-2}}\phi(z_{n-1})\right)$ 

• This gives (Zi, \$(Zi)), i=1,-,n

use polynomial interp to get Zn+1:  $\mathcal{P}(\phi(z_{n+1})) = \mathcal{P}(0) = z_{n+1}$ Terror

Downsides · Computationally expensive · Requires Ø(Z) to have differen inverse neare root (So it need root to be simple)