Boundary Value Problems (BVP's)

So far we've been solving IVP's like

$$\begin{cases} y' = f(y,t) \\ y(t_0) = y_0 \end{cases} \quad \& \quad \begin{cases} y'' = f(y', y, t) \\ y(t_0) = \alpha, \ y'(t_0) = \beta \end{cases}$$

In contrast, BVP's are differential equations with conditions at two different points.

For example:

$$\begin{cases} y'' = f(t, y, y') \\ y(a) = \alpha, \quad y(b) = \beta \end{cases}$$

Remark:

In general, we need 2 values to solve a 2^{nd} order ODE

Example: $y'' = -y$

has the general solution

$y(t) = A\sin t + B\cos t$

Then, given say $y(0)$ & $y'(0)$ as in IVP's we can solve for A, B

But we can also solve for A, B given $y(0)$ & $y(\pi/2)$ (say) (as in BVP's) assuming we get a consistent system

In general, we need numerical methods for solving BVP's!

We will soon consider these, but first let's worry about existence of solutions to BVP's

Theorem

Theorem: The BVP

$$\begin{cases} x'' = f(t,x) \\ x(0) = 0, \quad x(1) = 0 \end{cases}$$

has a unique sol'n if $\frac{\partial f}{\partial x}$ is cont., non-neg., bounded in $[0,1] \times \mathbb{R}$

Example: Show that

$$\begin{cases} x'' = (5x + \sin 3x)e^t \\ x(0) = x(1) = 0 \end{cases}$$

has a unique sol'n:

Sol'n: $f(t,x) = (5x + \sin 3x)e^t$

$$\Rightarrow \frac{\partial f}{\partial x} = (5 + 3\cos 3x)e^t$$

which is cont., non-neg (why?), & bounded (by $8e$) on $[0,1] \times \mathbb{R} \Rightarrow$ we have a unique sol'n by the theorem.

Applying the theorem to more general BVP's, like

$$\begin{cases} x'' = f(t,x) \\ x(a) = \alpha, \quad y(b) = \beta \end{cases}$$

We'll get there in 2 steps:

1st Theorem on two-point BVP's

Consider:

① $\begin{cases} x'' = f(t,x) \\ x(a) = \alpha, \quad x(b) = \beta \end{cases}$

② $\begin{cases} y'' = g(t,y) \\ y(0) = \alpha, \ y(1) = \beta \end{cases}$

where $g(p,q) = (b-a)^2 \left(f(a+(b-a)p, q \right)$

* If y solves ②, then

$$x(t) = y\left(\frac{t-a}{b-a}\right) \text{ solves ①}$$

* If x solves ①, then

$$y(t) = x(a+(b-a)t) \text{ solves ②.}$$

Proof: Simply check that the change of variables works (see book).

Remark: The above theorem simply remaps the interval $[a, b]$ to $[0, 1]$

Remark: To be able to use Theorem 0, we still have to deal with the fact that we have:

$$\begin{cases} y'' = g(t, y) \\ y(0) = \alpha, \quad y(1) = \beta \end{cases}$$

while we'd like to have $\alpha = \beta = 0$

Sol'n: Just subtract a linear function!

2^{nd} Theorem on two-point BVP's

Consider:

$$(2)\begin{cases} y'' = g(t,y) \\ y(0)=\alpha,\ y(1)=\beta \end{cases} \qquad (3)\begin{cases} z'' = h(t,z) \\ z(0)=0,\ z(1)=1 \end{cases}$$

where $h(p,q) = g(p, q+\alpha+(\beta-\alpha)p)$

* If z solves (3), then
  $y(t) = z(t) + \alpha + (\beta-\alpha)t$ solves (2)

* If y solves (2), then
  $z(t) = y(t) - [\alpha + (\beta-\alpha)t]$ solves (3).

Proof: Check.

Example on how to use the 3 theorems together

Convert the two-point BVP to an equivalent one with 0 boundary values on $[0,1]$

$$\begin{cases} x'' = x^2 + 3 - t^2 + xt \\ x(\underset{a}{3}) = 7 \quad , \quad x(\underset{b}{5}) = 9 \end{cases} \quad \text{—} \; (*)$$

Sol'n :

(1) By the 1st theorem :

$(*)$ is equivalent to

$$\begin{cases} y'' = g(t,y) \\ y(0) = 7 \, , \quad y(1) = 9 \end{cases}$$

with $g(t,y) = \underset{(b-a)^2}{4} f(\underbrace{3+2t}_{a+(b-a)t}, y)$

so $g(t,y) = 4\left[y^2 + 3 - (3+2t)^2 + x(3+2t)\right]$

(2) By the 2^{nd} theorem

$(*)$ is now equivalent to

$$z'' = h(t, z)$$
$$z(0) = 0 \quad , \quad z(1) = 0$$

with $\quad h(t, z) = g(t, z+7+2t)$

$$= 4\left[(z+7+2t)^2 + 3 - (3+2t)^2 + (z+7+2t)(3+2t)\right]$$

So, to solve for x, we can first solve for z, then substitute

$$y(t) = z(t) + 7 + 2t$$

and $\quad x(t) = y\left(\frac{t-3}{2}\right)$

$\boxtimes$

Theorem

Let $f(t,s)$ be cont's on $[0,1] \times \mathbb{R}$

with $|f(t,s_1) - f(t,s_2)| \leq k|s_1 - s_2|$

where $k < 8$

lipschitz cond'n

Then $\begin{cases} x'' = f(t,x) \\ x(0) = 0 = x(1) \end{cases}$

has a unique sol'n in $C[0,1]$

Example: $\begin{cases} x'' = 2e^{t\cos x} \\ x(0) = x(1) = 0 \end{cases}$

- $f(t,x) = 2e^{t\cos x}$

$\Rightarrow \dfrac{\partial f}{\partial x} = -2t\sin x \, e^{t\cos x}$

$\Rightarrow \left|\dfrac{\partial f}{\partial x}\right| \leq 2e$ on $[0,1] \times \mathbb{R}$

,

$f(t,x)$ is Lipschitz in x on $[0,1]\times\mathbb{R}$ with const $2e < 8$ $\Rightarrow$ we have a unique sol'n by the theorem above.

Solving BVP's: Shooting Methods

$$\begin{cases} x'' = f(t,x,x') \\ x(a) = \alpha \ , \ x(b) = \beta \end{cases}$$

Rough Idea:

(1) Convert the BVP to an IVP by guessing the value of $x'(a)$

(2) Integrate the equation

(3) Check if $x(b)=\beta$. If $x(b)\neq\beta$, guess $x'(a)$ and try again

So now:

$$(*) \begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \qquad x'(a) = z \end{cases}$$

this is a guess

Let the sol'n of $(*)$ be x_z.
We'd like $x_z = \beta$, so we

define $\phi(z) = x_z(b) - \beta$

the error from our guess

Our goal is to get $\phi(z) = 0$.

This is "just" a nonlinear equation that we need to solve

$\Rightarrow$ can use Newton's method, secant method, bisection method, etc....

OTOH, we don't explicitly have $\phi(z)$. Need to solve an IVP numerically for evaluation of $\phi(z)$

Secant Method

Recall: to solve $\phi(z) = 0$

we run the iteration

$$z_n = z_{n-1} - \left(\frac{z_{n-1} - z_{n-2}}{\phi(z_{n-1}) - \phi(z_{n-2})} \right) \phi(z_{n-1})$$

Now, after n iterations, we have $(z_i, \phi(z_i))$ for $i = 1, \dots, n$

$\Rightarrow$ can use polynomial interp to get a polynomial with

$$P(\phi(z_i)) = z_i$$

approximation of ϕ^{-1} by P!

now let $z_{n+1} = P(0)$

$\phi(z_{n+1}) = 0$

Use z_{n+1} as the next guess for $x'(a) = z$.

To summarize:

- Want to solve $\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \; x(b) = \beta \end{cases}$

- Instead, we solve
$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \; x'(a) = z_1 \end{cases}$$

- This give a sol'n x_{z_1} & a boundary value $x_{z_1}(b)$

So our error is

$$\phi(z_1) = x_{z_1}(b) - \beta$$

- Do it again with z_2 to get

$$\phi(z_2) = x_{z_2}(b) - \beta$$

- Now run the secant method

$$\Rightarrow \quad z_n = z_{n-1} - \left(\frac{\phi(z_{n-1}) - \phi(z_{n-2})}{z_{n-1} - z_{n-2}}\right)\phi(z_{n-1})$$

- This gives $(z_i, \phi(z_i)),\ i=1,\dots,n$

use polynomial interp to get z_{n+1}:

$$P(\phi(z_{n+1})) = P(0) = z_{n+1}$$

error

Downsides

- Computationally expensive
- Requires $\phi(z)$ to have differen inverse near root
  (so it need root to be simple)